forward and reverse propagation. In the figure it is shown the attenuation for the empty guide, *i.e.*, for the same guide without the slab of ferrite.

Let's follow the forward propagation experimental curve from left to right. When the slab of ferrite is against the left side wall the attenuation is at a minimum. By moving the slab of ferrite away from the wall, after a region of low attenuation, the signal goes very rapidly below the level of the attenuation of the empty guide. This is justified by the theoretical results of Fig. 4 and by the considerations about the group velocity. In fact, beyond the distance b_{0c}' no propagation can exist and below such a distance always exists a propagating mode with group velocity in the forward direction.

Moving the slab further away from the left side wall, the signal remains at a level below the empty guide attenuation level until we reach a distance approximately equal to $b-b_f-b_{0c}'$ when the attenuation begins to decrease and then, after it has reached a minimum, increases again above the empty guide attenuation. This last behavior is easily explained by the theoretical results, since beyond the distance $b-b_f-b_{0c}'$, energy begins to pass in the forward direction through modes having positive group velocity. However, since these modes have propagation constants going to $-\infty$ as the slab of ferrite approaches the right-hand side wall and cannot therefore allow propagation in the limit, the minimum of the attenuation experimentally found is explained.

From the preceding discussion we can conclude that there is good agreement between theory and experiments.

Higher-Order Evaluation of Electromagnetic Diffraction by Circular Disks*

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Summary—**The problem of the diffraction of an arbitrary electromagnetic field by a circular perfectly-conducting disk has been solved by using a series representation in powers of $k=2\pi/\lambda$ and the rectangular disk coordinates. The surface current density is given in terms of the field and its derivatives at the center of the disk. General expressions for the electric- and magnetic-dipole moments, the far-field and the scattering coefficient for the case of a plane wave at arbitrary incidence are presented. The calculations agree with results published by other authors. A bibliography of the most recent publications on this problem is included.**

I. Introduction

The problem of the diffraction by a circular conducting disk (or the complementary problem for a circular aperture in an infinite plane conducting screen) has occupied many workers in the field of diffraction theory. The problem can be formulated as follows:

1) the electromagnetic field has to obey Maxwell's equations,
2) the boundary conditions on the surface of the disk have to be fufilled, *e.g.*, for a perfectly-conducting disk the tangential electric field must vanish,
3) the edge conditions [48] at the rim of the disk have to be obeyed; they require that the field energy remains finite, or that the energy density has to be integrable over any finite space. This leads to the requirement that the normal component of the electric field increases not faster than $(1/r)^{1/2}$ where r is the distance from the edge,
4) Sommerfeld's radiation conditions [47] have to be fulfilled.

In this paper a power-series solution in (ka) valid for the small disk problem ($a<\lambda/2\pi$, where a = disk radius, λ = free-space wavelength) and an arbitrary incident field is given. It is essentially an extension of a procedure described by Bouwkamp [45]. The surface current density on the disk up to the third-order approximation in (ka) is calculated in terms of the electromagnetic field and its derivatives at the center of the disk. From these results expressions for the induced electric and magnetic dipole moments and the far-zone fields are derived. The scattering coefficient for a plane wave at arbitrary incidence has been calculated in agreement with formulas given by Lur'e [19] and Kuritsyn [20]. The essential advantage of the expressions obtained in this paper is that they can be used for any primary field. This is important in the case where interaction between several disks is considered. If the spacing between the disks is not large compared with the wavelength, the interaction fields cannot be approximated by a plane wave and the interaction be-

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between the induced dipole moments and higher-order multipole moments have to be taken into account. Some of this work will be reported at a later date.

II. Power-Series Solution

General Formulation

We shall consider a perfectly-conducting circular disk of radius a with its axis along the z direction as in Fig. 1. The magnetic vector potential $\boldsymbol{A}(\boldsymbol{R})$ for the diffracted field is given by

$$\boldsymbol{A}(\boldsymbol{R}) = \frac{\mu_0}{4\pi}\int_D J(\boldsymbol{p}')\frac{e^{-jkr}}{r}\,dS' \tag{1}$$

where $J(\boldsymbol{p}')$ is the electric surface current density and the integration is carried out over the disk D. $\boldsymbol{R}$ and $\boldsymbol{p}'$ are the coordinates of the field point $P(x, y, z)$ and the surface element dS' respectively, while r is the radius vector from dS' to $P(x, y, z)$. The time dependence $e^{j\omega t}$ is omitted throughout. The scattered field is found from the following relations:

$$\boldsymbol{H} = \frac{1}{\mu_0}\nabla\times\boldsymbol{A} \tag{2}$$

$$\boldsymbol{E} = -j\omega\boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\mu_0\epsilon_0}\cdot \tag{3}$$

The boundary conditions are satisfied if

$$\begin{aligned} E_x(\boldsymbol{R}) &= -E_x^i(\boldsymbol{R}) \\ E_y(\boldsymbol{R}) &= -E_y^i(\boldsymbol{R}) \quad \text{on the disk} \\ H_z(\boldsymbol{R}) &= -H_z^i(\boldsymbol{R}) \end{aligned} \tag{4}$$

where the superscript i indicates the incident field. We now express (2) and (3) in rectangular coordinates and combine them with (4)

$$H_z = \frac{1}{\mu_0}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = -H_z^i \tag{5}$$

$$\begin{aligned} E_x &= -j\omega A_x + \frac{1}{j\omega\mu_0\epsilon_0}\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right) \\ &= -j\omega A_x + \frac{1}{j\omega\mu_0\epsilon_0}\left[\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial y} - \mu_0 H_z^i\right)\right] \\ &= -j\omega A_x + \frac{1}{j\omega\epsilon_0\mu_0}\left[\nabla_{xy}^2 A_x - \mu_0\frac{\partial H_y^i}{\partial z} - j\omega\mu_0\epsilon_0 E_x^i\right] \\ &= -E_x^i \end{aligned} \tag{6}$$

$\nabla_{xy}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is defined as the two-dimensional Laplace operator.

In (6) we used (5) and Maxwell's equations for E_x^i. A similar expression as (6) is obtained for E_y. These equations can now be written in the form

$$\begin{aligned} \nabla_{xy}^2 A_x + k^2 A_x &= \mu_0\frac{\partial H_y^i}{\partial z} \\ \nabla_{xy}^2 A_y + k^2 A_y &= -\mu_0\frac{\partial H_x^i}{\partial z} \quad \text{on the disk} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= -\mu_0 H_z^i \end{aligned} \tag{7}$$

where $k = \omega(\mu_0\epsilon_0)^{1/2}$ is the free-space wave number.

Eqs. (7) are the basic differential equations which we shall solve for the vector potential $\boldsymbol{A}(x, y)$ on the disk. Knowing $\boldsymbol{A}(x, y)$ we then obtain the surface-current distribution $J(x', y')$ from the integral equation (1).

In order to obtain a unique solution the conditions 1)–4) in Section I have to be fulfilled. Conditions 1) and 2) have been taken care of in (2)–(4). The edge condition 3) will be fulfilled by assuming a suitable current distribution on the disk, and the radiation conditions 4) are secured by (1) for any finite current distribution.

In the following we attempt to find a power-series solution for the current J in terms of (ka), which is expected to converge well for small disks where $(ka) = 2\pi(a/\lambda) < 1$ or $a < \lambda/2\pi$ (λ = free-space wavelength).

First, we consider an expansion of $\boldsymbol{A}$, J and the right-hand side of (7) in powers of (k). We obtain the following expressions:

$$J = J^0 + kJ^1 + k^2J^2 + k^3J^3 + k^4J^4 + \cdots \tag{8}$$

$$\begin{aligned} Je^{-jkr} = {} & J^0 + k(J^1 - jrJ^0) + k^2(J^2 - jrJ^1 - 1/2r^2J^0) \\ & + k^3(J^3 - jrJ^2 - 1/2r^2J^1 + 1/6jr^3J^0) \\ & + k^4(J^4 - jrJ^3 - 1/2r^2J^2 + 1/6jr^3J^1 \\ & + 1/24r^4J^0) + \cdots \end{aligned} \tag{9}$$

$$\boldsymbol{A} = \boldsymbol{A}^0 + k\boldsymbol{A}^1 + k^2\boldsymbol{A}^2 + k^3\boldsymbol{A}^3 + k^4\boldsymbol{A}^4 + \cdots \tag{10}$$

where

$$\begin{aligned} \boldsymbol{A}^0 &= \frac{\mu_0}{4\pi}\int_D J^0\frac{dS}{r} \\ \boldsymbol{A}^1 &= \frac{\mu_0}{4\pi}\left\{\int_D J^1\frac{dS}{r} - j\int_D J^0 dS\right\} \\ \boldsymbol{A}^2 &= \frac{\mu_0}{4\pi}\left\{\int_D J^2\frac{dS}{r} - j\int_D J^1 dS - 1/2\int_D J^0 r\,dS\right\} \\ \boldsymbol{A}^3 &= \frac{\mu_0}{4\pi}\left\{\int_D J^3\frac{dS}{r} - j\int_D J^2 dS - 1/2\int_D J^1 r\,dS + 1/6j\int_D J^0 r^2 dS\right\} \\ \boldsymbol{A}^4 &= \frac{\mu_0}{4\pi}\left\{\int_D J^4\frac{dS}{r} - j\int_D J^3 dS - 1/2\int_D J^2 r\,dS + 1/6j\int_D J^1 r^2 dS + 1/24j\int_D J^0 r^3 dS\right\}. \end{aligned} \tag{11}$$

For simplification we adopt the following notation:

$$\begin{aligned}
\boldsymbol{A}^0 &= \boldsymbol{A}^{00} \\
\boldsymbol{A}^1 &= \boldsymbol{A}^{11} - j\boldsymbol{A}^{01} \\
\boldsymbol{A}^2 &= \boldsymbol{A}^{22} - j\boldsymbol{A}^{12} - 1/2\boldsymbol{A}^{02} \\
\boldsymbol{A}^3 &= \boldsymbol{A}^{33} - j\boldsymbol{A}^{23} - 1/2\boldsymbol{A}^{13} + 1/6j\boldsymbol{A}^{03} \\
\boldsymbol{A}^4 &= \boldsymbol{A}^{44} - j\boldsymbol{A}^{34} - 1/2\boldsymbol{A}^{24} + 1/6j\boldsymbol{A}^{14} + 1/24\boldsymbol{A}^{04} \quad (12)
\end{aligned}$$

where the partial vector potentials $\boldsymbol{A}^{mn}$ are defined by (11) and (12).

$$\begin{aligned}
\mu_0 \frac{\partial H_y^i}{\partial z} &= S^0 + kS^1 + k^2S^2 + k^3S^3 + \cdots \\
-\mu_0 \frac{\partial H_x^i}{\partial z} &= T^0 + kT^1 + k^2T^2 + k^3T^3 + \cdots \\
-\mu_0 H_z &= U^0 + kU^1 + k^2U^2 + k^3U^3 + \cdots . \quad (13)
\end{aligned}$$

The coefficients S^n, T^n, and U^n in (13) are, of course, functions of the field coordinates x and y and can, therefore, be written as

$$\begin{aligned}
S^n &= S_0^n + S_1^n x + S_2^n y + S_3^n x^2 + S_4^n xy \\
&\quad + S_5^n y^2 + \cdots \\
T^n &= T_0^n + T_1^n y + T_2^n x + T_3^n y^2 + T_4^n yx \\
&\quad + T_5^n x^2 + \cdots \\
U^n &= U_0^n + U_1^n x + U_2^n y + U_3^n x^2 + U_4^n xy \\
&\quad + U_5^n y^2 + \cdots . \quad (14)
\end{aligned}$$

The coefficients S_m^n, T_m^n and U_m^n are constants and can be calculated from the primary field. For the term $k^n x^r y^s$ we obtain

$$\begin{aligned}
S_m^n &= \frac{1}{n!r!s!} \frac{\partial^{r+s}}{\partial x^r \partial y^s} \frac{\partial^n}{\partial k^n} \left[\mu_0 \left(\frac{\partial H_y^i}{\partial z} \right) \right]_{\substack{x=y=z=0 \\ k=0}} \\
T_m^n &= \frac{1}{n!r!s!} \frac{\partial^{r+s}}{\partial x^r \partial y^s} \frac{\partial^n}{\partial k^n} \left[-\mu_0 \left(\frac{\partial H_y^i}{\partial z} \right) \right]_{\substack{x=y=z=0 \\ k=0}} \\
U_m^n &= \frac{1}{n!r!s!} \frac{\partial^{r+s}}{\partial x^r \partial y^s} \frac{\partial^n}{\partial_k n} \left[-\mu_0 H_z^i \right]_{\substack{x=y=z=0 \\ k=0}} . \quad (15)
\end{aligned}$$

Where all derivatives are taken for $x=y=z=k=0$. The subscript m can be found by inspection of (14) or from the relation

$$m = \left(\sum_{t=1}^{r+s} t \right) + s. \quad (15a)$$

We now substitute (10) and (13) in the basic equation (7). Equating equal powers of k we obtain for the nth-order approximation, *i.e.*, for k^n, the following relations:

$$\begin{aligned}
\nabla^2 A_x^n + A_x^{n-2} &= S^n \\
\nabla^2 A_y^n + A_y^{n-2} &= T^n \\
\frac{\partial A_y^n}{\partial x} - \frac{\partial A_x^n}{\partial y} &= U^n . \quad (16)
\end{aligned}$$

It is interesting to note that $\boldsymbol{A}^n$ does depend on the value of $\boldsymbol{A}^{n-2}$. Consider now (16) in terms of the partial vector potentials as defined in (11) and (12). It is readily seen that the second term in these expressions, $\boldsymbol{A}^{(n-1)n}$, is independent of the field coordinates x and y, so that all its derivatives vanish identically. Summarizing we find that in the calculation for $\boldsymbol{A}^n$ the expressions for $\boldsymbol{A}^{n-1}$ and $\boldsymbol{A}^{(n-1)n}$ or, in terms of the surface currents, the term J^{n-1}, do not appear. From this follows that the zero- and first-order approximation for J are not related to each other and can be evaluated from the expansion coefficients of the primary field only. For higher-order approximations, however, we have to use the results of the previous calculations. This can be clearly seen, if we rewrite (16) and keep only the term $\boldsymbol{A}^{nn}$ on the left-hand side. We then obtain

$$\nabla^2 A_x^{nn} = V^n \quad (17a)$$

$$\nabla^2 A_y^{nn} = W^n \quad (17b)$$

$$\frac{\partial A_y^{nn}}{\partial x} - \frac{\partial A_x^{nn}}{\partial y} = X^n \quad (17c)$$

where V^n, W^n and X^n are functions of the field coordinates x and y and can be written as follows:

$$\begin{aligned}
V^n &= V_0^n + V_1^n x + V_2^n y + V_3^n x^2 + V_4^n xy \\
&\quad + V_5^n y^2 + \cdots = \sum_{r,s} V_m^n x^r y^s \\
W^n &= W_0^n + W_1^n y + W_2^n x + W_3^n y^2 + W_4^n yx \\
&\quad + W_5^n x^2 + \cdots = \sum_{r,s} W_m^n y^r x^s \\
X^n &= X_0^n + X_1^n x + X_2^n y + X_3^n x^2 + X_4^n xy \\
&\quad + X_5^n y^2 + \cdots = \sum_{r,s} X_m^n x^r y^s . \quad (18)
\end{aligned}$$

Again m is found from (15a).

V_m^n, W_m^n and X_m^n can be calculated from the coefficients S_m^n, T_m^n and U_m^n for the primary field and from the results of the $(n-2)$ and lower-order approximations as will be shown explicitly in (29). The A_x^{nn} and A_y^{nn} can then be obtained from (17) by straightforward integration in form of power series in x and y. The main problem is now to find a solution for the integral equation for the current J^n. This equation is obtained from (11) and (12):

$$\boldsymbol{A}^{nn}(x, y) = \frac{\mu_0}{4\pi} \int_D J^n(x', y') \frac{dx'dy'}{[(x - x')^2 + (y - y')^2]^{1/2}} . \quad (19)$$

Formal Solution for the Surface Current Distribution

The kind of integral which appears in (19) has been investigated by Bouwkamp [45], [49], [50]. He found that if we write the current distribution in the following form:

$$\frac{\mu_0}{4\pi} J_x(x', y') = \frac{f_x(x', y')}{\pi^2(a^2 - \rho'^2)^{1/2}}$$
$$\frac{\mu_0}{4\pi} J_y(x', y') = \frac{f_y(x', y')}{\pi^2(a^2 - \rho'^2)^{1/2}}$$
$$(\rho')^2 = (x')^2 + (y')^2 \tag{20}$$

where f_x and f_y are polynomials in x' and y', the integral in (19) is also a polynomial in x and y, and is of the same order as $f(x', y')$. Table I in the Appendix gives a solution for the following integral:

$$G(x, y) = \frac{1}{\pi^2}\int_D \frac{f(x', y')dS}{(a^2 - \rho'^2)^{1/2}[(x - x')^2 + (y - y')^2]^{1/2}}. \tag{21}$$

The integration is over the surface D of the disk; $f(x', y')$ is a polynomial in x' and y'. Only terms up to the fourth power in x' and y' have been calculated. Computations of higher-order terms are possible yet they become, in principle, extremely tedious. It is noteworthy that $G(x, y)$ is also a polynomial of the same order as $f(x', y')$.

In order to obtain a unique solution, we have to ascertain that the edge conditions are fulfilled. It can be shown that they are equivalent to the requirement that the current component normal to the edge vanishes at the rim of the disk. The radial current is

$$J_r = J_x \cos\varphi' + J_y \sin\varphi' = (1/\rho')(x'J_x + y'J_y)$$

and thus we must have

$$\frac{4}{\pi\mu_0} \lim_{\rho'\to a} \frac{x'f_x + y'f_y}{\rho'(a^2 - \rho'^2)^{1/2}} = 0. \tag{22}$$

We further require that the total charge on the disk remains finite. The charge density σ is given by

$$\sigma = \frac{j}{\omega}\nabla\cdot J = \frac{j}{\omega}\frac{4}{\pi\mu_0}\nabla\cdot\frac{f}{(a^2 - \rho'^2)^{1/2}}$$
$$= \frac{4j}{\omega\pi\mu_0}\left[\left(\frac{\partial f_x}{\partial x'} + \frac{\partial f_y}{\partial y'}\right)\frac{1}{(a^2 - \rho'^2)^{1/2}} + \frac{x'f_x + y'f_x}{(a^2 - \rho'^2)^{3/2}}\right]. \tag{23}$$

The last term in (23) is only integrable over the disk if

$$(x'f_x + y'f_y) = D(x', y')(a^2 - \rho'^2) \tag{24}$$

where $D(x', y')$ is also a polynomial in x' and y' as will be shown in the following calculations. This condition, however, is sufficient to satisfy (22) and is, therefore, equivalent to the edge condition.

The current-distribution function is now expanded in terms of k as follows:

$$f = f^0 + kf^1 + k^2f^2 + k^3f^3 + \cdots. \tag{25}$$

We then write for the polynomial

$$f^n(x', y') = f_x^n(x', y')a_x + f_y^n(x', y')a_y$$
$$f_x^n(x', y') = a_0^n + a_1^n x' + a_2^n y' + a_3^n x'^2 + a_4^n x'y' + a_5^n y'^2 + a_6^n x'^3 + a_7^n x'^2y' + a_8^n x'y'^2 + a_9^n y'^3 + a_{10}^n x'^4 + a_{11}^n x'^3y' + a_{12}^n x'^2y'^2 + a_{13}^n x'y'^3 + a_{14}^n y'^4 + \cdots = \sum_{r,s} a_m^n x^r y^s \tag{26a}$$
$$f_y^n(x', y') = b_0^n + b_1^n y' + b_2^n x' + b_3^n y'^2 + b_4^n y'x' + b_5^n x'^2 + b_6^n y'^3 + b_7^n y'^2x' + b_8^n y'x'^2 + b_9^n x'^3 + b_{10}^n y'^4 + b_{11}^n y'^3x' + b_{12}^n y'^2x'^2 + b_{13}^n y'x'^3 + b_{14}^n x'^4 + \cdots = \sum_{r,s} b_m^n y^r x^s. \tag{26b}$$

The edge conditions (24) become:

$$(x'f_x^n + y'f_y^n) = (d_0^n + d_1^n x' + d_2^n y' + d_3^n x'^2 + d_4^n x'y' + d_5^n y'^2 + d_6^n x'^3 + d_7^n x'^2y' + d_8^n x'y'^2 + d_9^n y'^3)(a^2 - x'^2 - y'^2). \tag{27}$$

Calculations of the Current Coefficients a and b

Eqs. (17)–(20), (26), and (27) determine now a total of 40 coefficients a^n and b^n, if in $f(x'y')$ terms up to the fourth power in x' and y' are considered. Detailed calculations are presented in a separate report [52]. Here a short outline of the procedure is given: the coefficients a_m^n and b_m^n are found in terms of V_m^n, W_m^n, and X_m^n defined in (17) and (18)

$$a_m^n = a_m^n(V_{m'}^n, W_{m'}^n, X_{m'}^n)$$
$$b_m^n = b_m^n(V_{m'}^n, W_{m'}^n, X_{m'}^n). \tag{28}$$

The prime on the subscripts m in the brackets indicate that terms with a different subscript than m are involved.

Our next problem consists in expressing V_m^n, W_m^n, and X_m^n in terms of the field quantities S_m^n, T_m^n, and U_m^n. If we use the partial vector potentials given in (11), (12), and (16) and keep only the terms with A^n on the left-hand side, we obtain together with (17), (20) and (26)

$$V_m^n = V_m^n(S_m^n, a_{m'}^{n-2}, a_{m'}^{n-3}, \cdots)$$
$$W_m^n = W_m^n(T_m^n, b_{m'}^{n-2}, b_{m'}^{n-3}, \cdots)$$
$$X_m^n = X_m^n(U_m^n, a_{m'}^{n-2}, a_{m'}^{n-3}, \cdots, b_{m'}^{n-2}, b_{m'}^{n-3}, \cdots) \tag{29}$$

where only current coefficients of $(n-2)$ and lower order are involved. Putting these expressions back in (28) and replacing all the current coefficients of lower order by an iterative process we finally obtain an expression which involves only the field coefficients S_m^n, T_m^n, and U_m^n

$$a_m^n = a_m^n(S_{m'}^{n'}, T_{m'}^{n'}, U_{m'}^{n'})$$
$$b_m^n = b_m^n(S_{m'}^{n'}, T_{m'}^{n'}, U_{m'}^{n'}). \tag{30}$$

Following (25) we define the total current coefficients by

$$a_m = a_m^0 + ka_m^1 + k^2a_m^2 + k^3a_m^3 + \cdots$$
$$b_m = b_m^0 + kb_m^1 + k^2b_m^2 + k^3b_m^3 + \cdots \quad (31)$$

and similarly for the expansion coefficients of the incident field

$$S_m = \sum_{n=0}^{\infty} k^n S_m^n = \sum_{n=0}^{\infty} \left\{ \frac{k^n}{n!} \frac{\partial^n}{\partial k^n} \left[\frac{\partial^{r+s}}{\partial x^r \partial y^s} \cdot \left(\mu_0 \frac{\partial H_y^i}{\partial z} \right) \right] \right\}_{x=y=z=k=0} \frac{1}{r!s!}$$
$$= \frac{1}{r!s!} \left[\frac{\partial^{r+s}}{\partial_x r \partial_y s} \left(\mu_0 \frac{\partial H_y^i}{\partial z} \right) \right]_{x=y=z=0}$$

$$T_m = \sum_{n=0}^{\infty} k^n T_m^n = \sum_{n=0}^{\infty} \left\{ \frac{k^n}{n!} \frac{\partial^n}{\partial k^n} \left[\frac{\partial^{r+s}}{\partial x^r \partial y^s} \cdot \left(-\mu_0 \frac{\partial H_x^i}{\partial z} \right) \right] \right\}_{x=y=z=k=0} \frac{1}{r!s!}$$
$$= \frac{1}{r!s!} \left[\frac{\partial^{r+s}}{\partial y^r \partial x^s} \left(-\mu_0 \frac{\partial H_x^i}{\partial z} \right) \right]_{x=y=z=0}$$

$$U_m = \sum_{n=0}^{\infty} k^n U_m^n = \sum_{n=0}^{\infty} \left\{ \frac{k^n}{n!} \frac{\partial^n}{\partial k^n} \left[\frac{\partial^{r+s}}{\partial y^r \partial x^s} \cdot (-\mu_0 H_z^i) \right] \right\}_{x=y=z=k=0} \frac{1}{r!s!}$$
$$= \frac{1}{r!s!} \left[\frac{\partial^{r+s}}{\partial y^r \partial x^s} (-\mu_0 Hz^i) \right]_{x=y=z=0}. \quad (32)$$

The right-hand side is obtained by using (15) and Taylor's expansion theorem.

It turns out that all current coefficients belonging to different powers of k^n but having the same subscript m have the same functional dependence on S_m^n, T_m^n, and U_m^n. This allows us to write the total current coefficients in terms of the primary field directly by using (32). After some calculations we finally obtain

$$a_0 = \frac{a^2}{315}\mu_0 \left\{ 210\left(-2\frac{\partial H_y}{\partial z} + \frac{\partial H_z}{\partial y} \right) + a^2\left(-28\frac{\partial^3 H_y}{\partial x^2 \partial z} - 28\frac{\partial^3 H_y}{\partial y^2 \partial z} + 21\frac{\partial^3 H_z}{\partial y^3} + 21\frac{\partial^3 H_z}{\partial x^2 \partial y} \right) + (ka)^2\left(-196\frac{\partial H_y}{\partial z} + 168\frac{\partial H_z}{\partial y} \right) \right\} + j\frac{8a^2}{9\pi}\mu_0 (ka)^3 \left(\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} \right)$$

$$a_1 = \frac{a^2}{30}\mu_0 \left\{ -22\frac{\partial^2 H_y}{\partial x \partial z} - 2\frac{\partial^2 H_x}{\partial y \partial z} + 8\frac{\partial^2 H_z}{\partial x \partial y} \right\}$$

$$a_2 = \frac{\mu_0}{30} \left\{ 30H_z + a^2\left(-13\frac{\partial^2 H_y}{\partial y \partial z} + 11\frac{\partial^2 H_x}{\partial x \partial z} - 8\frac{\partial^2 H_z}{\partial x^2} \right) - (ka)^2 H_z \right\} + j\frac{4a^2}{9\pi}\mu_0 (ka)^3 H_z$$

$$a_3 = \frac{\mu_0}{315} \left\{ 210\left(2\frac{\partial H_y}{\partial z} - \frac{\partial H_z}{\partial y} \right) + a^2\left(-52\frac{\partial^3 H_y}{\partial x^2 \partial z} + 44\frac{\partial^3 H_y}{\partial y^2 \partial z} - 9\frac{\partial^3 H_z}{\partial x^2 \partial y} - 33\frac{\partial^3 H_z}{\partial y^3} \right) + (ka)^2\left(224\frac{\partial H_y}{\partial z} - 195\frac{\partial H_z}{\partial y} \right) \right\} + j\frac{8\mu_0}{9\pi}(ka)^3\left(-\frac{\partial H_y}{\partial z} + \frac{\partial H_z}{\partial y} \right)$$

$$a_4 = \frac{\mu_0}{315} \left\{ 210\left(-\frac{\partial H_x}{\partial z} + 2\frac{\partial H_z}{\partial x} \right) + a^2\left(-27\frac{\partial^3 H_x}{\partial y^2 \partial z} + 165\frac{\partial^3 H_x}{\partial x^2 \partial z} - 150\frac{\partial^3 H_z}{\partial x^3} - 102\frac{\partial^3 H_z}{\partial x \partial y^2} \right) + (ka)^2\left(-168\frac{\partial H_x}{\partial z} - 12\frac{\partial H_z}{\partial x} \right) \right\} + j\frac{8\mu_0}{9\pi}(ka)^3\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right)$$

$$a_5 = \frac{\mu_0}{315} \left\{ 210\left(\frac{\partial H_y}{\partial z} + \frac{\partial H_z}{\partial y} \right) + a^2\left(17\frac{\partial^3 H_y}{\partial x^2 \partial z} - 79\frac{\partial^3 H_y}{\partial y^2 \partial z} + 9\frac{\partial^3 H_z}{\partial x^2 \partial y} + 33\frac{\partial^3 H_z}{\partial y^3} \right) + (ka)^2\left(56\frac{\partial H_y}{\partial z} - 15\frac{\partial H_z}{\partial y} \right) \right\}$$

$$a_6 = \frac{\mu_0}{30} \left\{ 22\frac{\partial^2 H_y}{\partial x \partial z} + 2\frac{\partial^2 H_x}{\partial y \partial z} - 8\frac{\partial^2 H_z}{\partial x \partial y} \right\}$$

$$a_7 = \frac{\mu_0}{30} \left\{ 14\frac{\partial^2 H_y}{\partial y \partial z} - 18\frac{\partial^2 H_x}{\partial x \partial z} + 24\frac{\partial^2 H_z}{\partial x^2} + 7k^2 H_z \right\}$$

$$a_8 = \frac{\mu_0}{30} \left\{ 14\frac{\partial^2 H_y}{\partial x \partial z} - 6\frac{\partial^2 H_x}{\partial y \partial z} + 24\frac{\partial^2 H_z}{\partial x \partial y} \right\}$$

$$a_9 = \frac{\mu_0}{30} \left\{ 22\frac{\partial^2 H_y}{\partial y \partial z} + 6\frac{\partial^2 H_x}{\partial x \partial z} - 8\frac{\partial^2 H_z}{\partial x^2} - 9k^2 H_z \right\}$$

$$a_{10} = \frac{\mu_0}{315} \left\{ 80\frac{\partial^3 H_y}{\partial x^2 \partial z} - 16\frac{\partial^3 H_y}{\partial y^2 \partial z} - 12\frac{\partial^3 H_z}{\partial x^2 \partial y} + 12\frac{\partial^3 H_z}{\partial y^3} + k^2\left(-28\frac{\partial H_y}{\partial z} + 27\frac{\partial H_z}{\partial y} \right) \right\}$$

$$a_{11} = \frac{\mu_0}{315} \left\{ 12\frac{\partial^3 H_x}{\partial y^2 \partial z} - 204\frac{\partial^3 H_x}{\partial x^2 \partial z} + 216\frac{\partial^3 H_z}{\partial x^3} + 120\frac{\partial^3 H_z}{\partial x \partial y^2} + k^2\left(21\frac{\partial H_x}{\partial z} + 150\frac{\partial H_z}{\partial x} \right) \right\}$$

$$a_{12} = \frac{\mu_0}{315}\left\{52\frac{\partial^3 H_y}{\partial x^2 \partial y} + 124\frac{\partial^3 H_y}{\partial y^2 \partial z} + 72\frac{\partial^3 H_z}{\partial x^2 \partial y} - 72\frac{\partial^3 H_z}{\partial y^3} + k^2\left(-35\frac{\partial H_y}{\partial z} - 36\frac{\partial H_z}{\partial y}\right)\right\}$$

$$a_{13} = \frac{\mu_0}{315}\left\{-12\frac{\partial^3 H_x}{\partial y^2 \partial z} - 132\frac{\partial^3 H_x}{\partial x^2 \partial z} + 120\frac{\partial^3 H_z}{\partial x^3} + 216\frac{\partial^3 H_z}{\partial x \partial y^2} + k^2\left(21\frac{\partial H_x}{\partial z} + 102\frac{\partial H_z}{\partial x}\right)\right\}$$

$$a_{14} = \frac{\mu_0}{315}\left\{-4\frac{\partial^3 H_y}{\partial x^2 \partial z} + 68\frac{\partial^3 H_y}{\partial y^2 \partial z} - 12\frac{\partial^3 H_z}{\partial x^2 \partial y} + 12\frac{\partial^3 H_z}{\partial y^3} + k^2\left(-7\frac{\partial H_y}{\partial z} - 15\frac{\partial H_z}{\partial y}\right)\right\}. \tag{33}$$

$$b_0 = \frac{a^2}{315}\mu_0\left\{210\left(2\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + a^2\left(28\frac{\partial^3 H_x}{\partial y^2 \partial z} + 28\frac{\partial^3 H_x}{\partial x^2 \partial z} - 21\frac{\partial^3 H_z}{\partial x^3} - 21\frac{\partial^3 H_z}{\partial x \partial y^2}\right) + (ka)^2\left(196\frac{\partial H_x}{\partial z} - 168\frac{\partial H_z}{\partial x}\right)\right\} + j\frac{8a^2}{9\pi}\mu_0(ka)^3\left(-\frac{\partial H_x}{\partial z} + \frac{\partial H_z}{\partial x}\right)$$

$$b_1 = \frac{a^2}{30}\mu_0\left\{22\frac{\partial^2 H_x}{\partial y \partial z} + 2\frac{\partial^2 H_y}{\partial x \partial z} - 8\frac{\partial^2 H_z}{\partial x \partial y}\right\}$$

$$b_2 = \frac{\mu_0}{30}\left\{-30H_z + a^2\left(13\frac{\partial^2 H_x}{\partial x \partial z} + 11\frac{\partial^2 H_y}{\partial y \partial z} + 8\frac{\partial^2 H_z}{\partial y^2}\right) + 9\,(ka)^2 H_z\right\} - j\frac{4a^2}{9\pi}\mu_0(ka)^3 H_z$$

$$b_3 = \frac{\mu_0}{315}\left\{210\left(-2\frac{\partial H_x}{\partial z} + \frac{\partial H_z}{\partial x}\right) + a^2\left(52\frac{\partial^3 H_x}{\partial y^2 \partial z} - 44\frac{\partial^3 H_x}{\partial x^2 \partial z} + 9\frac{\partial^3 H_z}{\partial x \partial y^2} + 33\frac{\partial^3 H_z}{\partial x^3}\right) + (ka)^2\left(-224\frac{\partial H_x}{\partial z} + 195\frac{\partial H_z}{\partial x}\right)\right\} + j\frac{8\mu_0}{9\pi}(ka)^3\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$b_4 = \frac{\mu_0}{315}\left\{210\left(\frac{\partial H_y}{\partial z} - 2\frac{\partial H_z}{\partial y}\right) + a^2\left(27\frac{\partial^3 H_y}{\partial x^2 \partial z} - 165\frac{\partial^3 H_y}{\partial y^2 \partial z} + 150\frac{\partial^3 H_z}{\partial y^3} + 102\frac{\partial^3 H_z}{\partial x^2 \partial y}\right) + (ka)^2\left(168\frac{\partial H_y}{\partial z} + 12\frac{\partial H_z}{\partial y}\right)\right\} + j\frac{8}{9\pi}\mu_0(ka)^3\left(-\frac{\partial H_y}{\partial z} + \frac{\partial H_z}{\partial y}\right)$$

$$b_5 = \frac{\mu_0}{315}\left\{210\left(-\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) + a^2\left(-17\frac{\partial^3 H_y}{\partial y^2 \partial z} + 79\frac{\partial^3 H_x}{\partial x^2 \partial z} - 9\frac{\partial^3 H_z}{\partial x \partial y^2} - 33\frac{\partial^3 H_z}{\partial x^3}\right) + (ka)^2\left(-56\frac{\partial H_x}{\partial z} + 15\frac{\partial H_z}{\partial x}\right)\right\}$$

$$b_6 = \frac{\mu_0}{30}\left\{-22\frac{\partial^2 H_x}{\partial y \partial z} - 2\frac{\partial^2 H_y}{\partial x \partial z} + 8\frac{\partial^2 H_z}{\partial x \partial y}\right\}$$

$$b_7 = \frac{\mu_0}{30}\left\{-14\frac{\partial^2 H_x}{\partial x \partial z} + 18\frac{\partial^2 H_y}{\partial y \partial z} - 24\frac{\partial^2 H_z}{\partial y^2} - 7k^2 H_z\right\}$$

$$b_8 = \frac{\mu_0}{30}\left\{-14\frac{\partial^2 H_x}{\partial y \partial z} + 6\frac{\partial^2 H_y}{\partial x \partial z} - 24\frac{\partial^2 H_z}{\partial x \partial y}\right\}$$

$$b_9 = \frac{\mu_0}{30}\left\{-22\frac{\partial^2 H_x}{\partial x \partial z} - 6\frac{\partial^2 H_y}{\partial y \partial z} + 8\frac{\partial^2 H_z}{\partial y^2} + 9k^2 H_z\right\}$$

$$b_{10} = \frac{\mu_0}{315}\left\{-80\frac{\partial^3 H_x}{\partial y^2 \partial z} + 16\frac{\partial^3 H_x}{\partial x^2 \partial z} + 12\frac{\partial^3 H_z}{\partial x \partial y^2} - 12\frac{\partial^3 H_z}{\partial x^3} + k^2\left(28\frac{\partial H_x}{\partial z} - 27\frac{\partial H_z}{\partial x}\right)\right\}$$

$$b_{11} = \frac{\mu_0}{315}\left\{-12\frac{\partial^3 H_y}{\partial x^2 \partial z} + 204\frac{\partial^3 H_y}{\partial y^2 \partial z} - 216\frac{\partial^3 H_z}{\partial y^3} - 120\frac{\partial^3 H_z}{\partial x^2 \partial y} + k^2\left(-21\frac{\partial H_y}{\partial z} - 150\frac{\partial H_z}{\partial y}\right)\right\}$$

$$b_{12} = \frac{\mu_0}{315}\left\{-52\frac{\partial^3 H_x}{\partial y^2 \partial z} - 124\frac{\partial^3 H_x}{\partial x^2 \partial z} - 72\frac{\partial^3 H_z}{\partial x \partial y^2} + 72\frac{\partial^3 H_z}{\partial x^3} + k^2\left(35\frac{\partial H_x}{\partial z} + 36\frac{\partial H_z}{\partial x}\right)\right\}$$

$$b_{13} = \frac{\mu_0}{315}\left\{12\frac{\partial^3 H_y}{\partial x^2 \partial z} + 132\frac{\partial^3 H_y}{\partial y^2 \partial z} - 120\frac{\partial^3 H_z}{\partial y^3} - 216\frac{\partial^3 H_z}{\partial x^2 \partial y} + k^2\left(-21\frac{\partial H_y}{\partial z} - 102\frac{\partial H_z}{\partial y}\right)\right\}$$

$$b_{14} = \frac{\mu_0}{315}\left\{4\frac{\partial^3 H_x}{\partial y^2 \partial z} - 68\frac{\partial^3 H_x}{\partial x^2 \partial z} + 12\frac{\partial^3 H_z}{\partial x \partial y^2} - 12\frac{\partial^3 H_z}{\partial x^3} + k^2\left(7\frac{\partial H_x}{\partial z} + 15\frac{\partial H_z}{\partial x}\right)\right\}. \tag{34}$$

The superscript i for the incident field has been omitted. All magnetic field components and their derivatives should, of course, be evaluated at the center of the disk.

Eqs. (33) and (34) represent the solution of our problem from which we shall derive the results in the following sections.

III. Induced Electric-Dipole Moment

The induced electric-dipole moment is defined by

$$P = \int_D p\sigma(p)dS \tag{35}$$

where

$$\sigma = \frac{j}{\omega}\nabla\cdot J \tag{36}$$

is the electric-surface-charge density and p the radius vector on the disk.

Obviously only odd-power terms in x' and y' for $\sigma(x', y')$ (or even-power terms for $J(x', y')$) contribute to $\boldsymbol{P}$.

Using the results in Section II and Maxwell's equation we obtain

$$P_x = \frac{16}{3}a^3\epsilon_0\left[E_x{}^i + \frac{(ka)^2}{30}\left(13E_x{}^i - \frac{3}{k^2}\frac{\partial^2 E_x{}^i}{\partial z^2} + \frac{2j}{\omega\epsilon_0}\frac{\partial H_z{}^i}{\partial y}\right) - \frac{8j}{9\pi}(ka)^3E_x{}^i\right]_0$$

$$P_y = \frac{16}{3}a^3\epsilon_0\left[E_y{}^i + \frac{(ka)^2}{30}\left(13E_y{}^i - \frac{3}{k^2}\frac{\partial^2 E_y{}^i}{\partial z^2} - \frac{2j}{\omega\epsilon_0}\frac{\partial H_z{}^i}{\partial x}\right) - \frac{8j}{9\pi}(ka)^3E_y{}^i\right]_0. \tag{37}$$

The subscript 0 at the right-hand side of the main bracket in (37) indicates that all terms have to be evaluated for $x=y=z=0$.

The first terms in (37) represent the well-known first-order approximations calculated by Bethe [21].

Eq. (37) shows also that in the first-order approximation $\boldsymbol{P}$ is in the direction of the tangential electric field at the center of the disk. The higher-order terms, however, lead to cross polarization, due to the terms $\partial H_z{}^i/\partial y$ and $\partial H_z{}^i/\partial x$.

For the case of an incident plane wave the electric-dipole moment depends also on the angle of incidence θ_i and is given by (see Fig. 1)

$$P_x = \frac{16}{3}a^3\epsilon_0\left[1 + \left(\frac{8}{15} - \frac{1}{10}\sin^2\theta_i\right)(ka)^2 - j\frac{8}{9\pi}(ka)^3\right]E_x{}^i(0, 0, 0)$$

$$P_y = \frac{16}{3}a^3\epsilon_0\left[1 + \left(\frac{8}{15} - \frac{1}{6}\sin^2\theta_i\right)(ka)^2 - j\frac{8}{9\pi}(ka)^3\right]E_y{}^i(0, 0, 0). \tag{38}$$

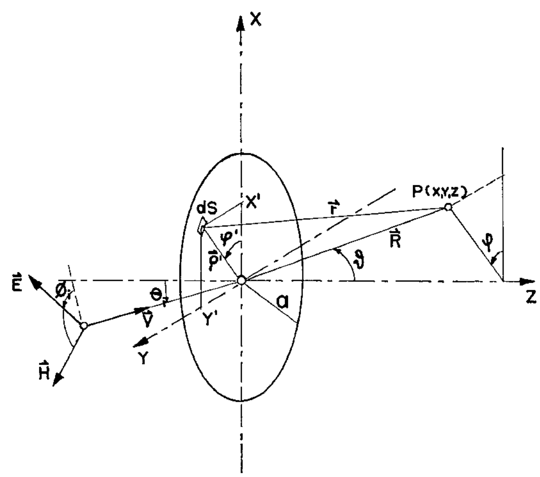

Fig. 1.

The case for normal incidence has been treated by Bouwkamp [23] up to the fifth-order approximation. From his current distribution, one obtains

$$\boldsymbol{P} = \frac{16}{3}a^3\epsilon_0\left[1 + \frac{8}{15}(ka)^2 - j\frac{8}{9\pi}(ka)^3 + \frac{16}{105}(ka)^4 - j\frac{176}{225\pi}(ka)^5\right]\boldsymbol{E}(0, 0, 0). \tag{39}$$

IV. Induced Magnetic-Dipole Moment

The magnetic-dipole moment is defined by

$$M = \frac{1}{2}\int_D p \times j(p)dS. \tag{40}$$

In the case of a plane disk $\boldsymbol{M}$ has only a component M_z along the disk axis. It is easily seen that only odd power terms in $J(x', y')$ contribute.

Again we can express M_z in terms of the primary field

$$M_z = -\frac{8a^3}{3}\left\{H_z{}^i - \frac{1}{10}(ka)^2\left[3H_z{}^i + \frac{1}{k^2}\frac{\partial^2 H_z{}^i}{\partial z^2}\right] + j\frac{4}{9\pi}(ka)^3H_z{}^i\right\}_0. \tag{41}$$

The bracketed term is evaluated at $x=y=z=0$.

For a plane wave incident at an angle θ_i (see Fig. 1), (41) becomes

$$M_z = -\frac{8}{3}a^3\left\{1 - \frac{1}{10}(2 + \sin^2\theta_i)(ka)^2 + j\frac{4}{9\pi}(ka)^3\right\}H_z{}^i(0, 0, 0). \tag{42}$$

For normal incidence the induced magnetic-dipole moment vanishes.

V. Far-Zone Fields

Given the current distribution we can find the scattered field in a straightforward manner from the vector potential as given in (1). Unfortunately a general solution of this integral is not available. It is, however, possible to evaluate it for field points which are at distances large compared with the disk radius. In this case we can write

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int_D J\frac{e^{-jkr}}{r}dS' \xrightarrow[R\to\infty]{} \frac{\mu_0}{4\pi}\frac{e^{-jkR}}{R} \cdot \int_D J\exp\left[jk\rho'\sin\vartheta\cos(\varphi-\varphi')\right]dS' \tag{43}$$

where R, ϑ, φ are the spherical coordinates of the field point, ρ' and φ' are the integration variables as shown in Fig. 1. Expanding the exponential in powers of k up to the term k^3 and using (8) one obtains an expression for $\boldsymbol{A}$ correct to the third order in k. We express the field in spherical coordinates

$$\begin{aligned} A_r &= A_x\cos\varphi\sin\vartheta + A_y\sin\varphi\sin\vartheta \\ A_\vartheta &= A_x\cos\varphi\cos\vartheta + A_y\sin\varphi\cos\vartheta \\ A_\phi &= -A_x\sin\varphi + A_y\cos\varphi. \end{aligned} \tag{44}$$

It turns out that for large distances the field components become

$$\begin{aligned} H_\vartheta &= -\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} & E_\varphi &= \frac{jk}{\mu_0}A_\varphi \\ H_\varphi &= \left(\frac{\epsilon_0}{\mu_0}\right)^{1/2} & E_\vartheta &= -\frac{jk}{\mu_0}A_\vartheta. \end{aligned} \tag{45}$$

After performing the straightforward but tedious computations one finally obtains in terms of the incident field

$$\begin{aligned} H_\vartheta = \frac{e^{-jkR}}{\pi R}(ka)^2a\Bigg\{&\frac{1}{4(\mu_0\epsilon_0)^{1/2}a^3}\left[-P_y\cos\varphi + P_x\sin\varphi\right] \\ &+ \sin\vartheta\Bigg[-\frac{1}{4a^3}M_z - \frac{4}{45}(ka)^2H_z^i \\ &+ j\frac{8}{45\omega\mu_0}(ka)^2\Bigg[-\frac{\partial E_x^i}{\partial y}\cos^2\varphi + \frac{\partial E_y^i}{\partial x}\sin^2\varphi \\ &+ \left(\frac{\partial E_x^i}{\partial x} - \frac{\partial E_y^i}{\partial y}\right)\sin\varphi\cos\varphi\Bigg]\Bigg] \\ &+ \frac{2}{45}(ka)^2\sin^2\vartheta\Bigg[\left(5\sqrt{\frac{\epsilon_0}{\mu_0}}E_y^i + j\frac{2}{k}\frac{\partial H_z^i}{\partial x}\right)\cos\varphi \\ &+ \left(-5\sqrt{\frac{\epsilon_0}{\mu_0}}E_x^i + j\frac{2}{k}\frac{\partial H_z^i}{\partial y}\right)\sin\varphi\Bigg] \\ &- \frac{1}{15}\sin^3\vartheta(ka)^2H_z^i\Bigg\}_0 \end{aligned} \tag{46a}$$

$$\begin{aligned} H_\varphi = \frac{e^{-jkR}}{\pi R}(ka)^2a\Bigg\{&\frac{1}{4(\mu_0\epsilon_0)^{1/2}a^3}\left[P_x\cos\varphi + P_y\sin\varphi\right] \\ &+ \frac{(ka)^2}{45}\Bigg[\frac{j}{\omega\mu_0}\sin\vartheta\Bigg[-6\frac{\partial E_z^i}{\partial z} - 8\frac{\partial E_y^i}{\partial y}\cos^2\varphi \\ &- 8\frac{\partial E_x^i}{\partial x}\sin^2\varphi + 8\left(\frac{\partial E_x^i}{\partial y} + \frac{\partial E_y^i}{\partial x}\right)\sin\varphi\cos\varphi\Bigg] \\ &+ 6\sqrt{\frac{\epsilon_0}{\mu_0}}\sin^2\vartheta\left[-E_x^i\cos\varphi\right. \\ &\left.- E_y^i\sin\varphi\right]\Bigg]\Bigg\}_0\cos\vartheta. \end{aligned} \tag{46b}$$

Again all the fields are evaluated for $x=y=z=0$.

For a plane wave at oblique incidence we obtain

$$\begin{aligned} H_\vartheta = \frac{e^{-jkR}}{\pi R}(ka)^2aH_0\Bigg\{&\frac{2}{3}\sin\phi_i\sin\theta_i\sin\vartheta \\ &+ \frac{4}{3}\cos\phi_i\cos\theta_i\sin\varphi - \frac{4}{3}\sin\phi_i\cos\varphi \\ &+ (ka)^2\frac{1}{45}\Bigg[(32 - 6\sin^2\theta_i)\cos\phi_i\cos\theta_i\sin\varphi \\ &- (32 - 10\sin^2\theta_i)\sin\phi_i\cos\varphi \\ &+ \sin\Big(8\cos\phi_i\cos\theta_i\sin\theta_i\sin\varphi\cos\varphi \\ &+ -(2 - 3\sin^2\theta_i - 8\cos^2\varphi)\sin\phi_i\sin\theta_i\Big) \\ &+ \sin^2\vartheta\Big(-10\cos\phi_i\cos\theta_i\sin\varphi \\ &+ (10 + 4\sin^2\theta_i)\sin\phi_i\cos\varphi\Big) \\ &- 3\sin^3\vartheta\sin\phi_i\sin\theta_i\Bigg]\Bigg\} \end{aligned} \tag{47a}$$

$$\begin{aligned} H_\varphi = \frac{e^{-jkR}}{\pi R}(ka)^2aH_0\cos\vartheta\Bigg\{&\frac{4}{3}(\cos\phi_i\cos\theta_i\cos\varphi \\ &+ \sin\phi_i\sin\varphi) \\ &+ (ka)^2\frac{1}{45}\Bigg[(32 - 6\sin^2\theta_i)\cos\phi_i\cos\theta_i\cos\varphi \\ &+ (32 - 10\sin^2\theta_i)\sin\phi_i\sin\varphi \\ &+ \sin\vartheta\Big((6 - 8\sin^2\varphi)\cos\phi_i\cos\theta_i\sin\theta_i \\ &+ 8\sin\phi_i\sin\theta_i\cos\varphi\sin\varphi\Big) \\ &- 6\sin^2\vartheta(\cos\phi_i\cos\theta_i\cos\varphi + \sin\phi_i\sin\varphi)\Bigg]\Bigg\}. \end{aligned} \tag{47b}$$

Here ϕ_i and θ_i are the angle of incidence as shown in Fig. 1. These results do not seem to agree with results obtained by Stevenson [24]. They lead, however, to the same expression for the scattering coefficient as obtained by Lur'e [19] and Kuritsyn [20].

VI. Scattering Coefficient for Incident Plane Wave

The scattering coefficient τ is defined as the ratio of the energy scattered from the disk to the energy incident on the disk. The scattered energy is found by integrating the Poynting vector of the scattered field over the sphere at infinity.

Using (47) one finds for an incident plane wave

$$\tau = \frac{\int_0^{2\pi}\int_0^{\pi} \mathrm{Re}\,(|H\vartheta|^2 + |H\phi|^2)\sin\vartheta d\vartheta d\varphi}{\pi a^2 H_0{}^2 \cos\theta_i}$$

$$= \frac{128}{27\pi^2\cos\theta_i}(ka)^4\left\{1 + \sin^2\theta_i\left(\frac{5}{4}\sin^2\phi_i - 1\right) + \frac{(ka)^2}{25}\left[(22 - 5\sin^2\theta_i)\cos^2\theta_i\cos^2\phi_i + \tfrac{1}{4}(88 - 54\sin^2\theta_i - 5\sin^4\theta_i)\sin^2\phi_i\right]\right\}. \quad (48)$$

For normal incidence Bouwkamp [23] obtained

$$\tau = \frac{128}{27\pi^2}(ka)^4\left[1 + \frac{22}{25}(ka)^2 + \frac{7512}{18375}(ka)^4 + 0(ka)^6\right]. \quad (49)$$

A general expression for τ for arbitrary primary fields would involve the separation of the real and imaginary parts of the incident fields and their derivatives at the center of the disk. The resulting expression would be very cumbersome and of little practical value.

VII. Diffraction by a Circular Aperture

Using the results of the generalized Babinet's principle [51] it can be shown that the disk problem and the aperture problem are equivalent if the following substitutions are made

substitute $\mu_0 \boldsymbol{H}^i{}_{\mathrm{Aperture}}$	for $\epsilon_0 \boldsymbol{E}^i{}_{\mathrm{Disk}}$
substitute $-\epsilon_0 \boldsymbol{E}^i{}_{\mathrm{Aperture}}$	for $\mu_0 \boldsymbol{H}^i{}_{\mathrm{Disk}}$
substitute $\mu_0 \boldsymbol{M}_{\mathrm{Aperture}}$	for $\boldsymbol{P}_{\mathrm{Disk}}$
substitute $\boldsymbol{P}_{\mathrm{Aperture}}$	for $\mu_0 \boldsymbol{M}_{\mathrm{Disk}}$. (50)

It is customary to designate as primary field $\boldsymbol{E}^0$, $\boldsymbol{H}^0$ the field which would exist if the aperture is replaced by a solid screen. We obtain thus

$$\boldsymbol{E}_z{}^i = \tfrac{1}{2}\boldsymbol{E}_z{}^0$$
$$\boldsymbol{H}_{\tan} = \tfrac{1}{2}\boldsymbol{H}_{\tan}{}^0. \quad (51)$$

This leads to the following expressions for the induced electric and magnetic dipole moments:

$$P_z = \frac{4}{3}a^3\epsilon_0\left\{E_z{}^0 - \frac{1}{10}(ka)^2\left[3E_z{}^0 + \frac{1}{k^2}\frac{\partial^2 E_z{}^0}{\partial z^2}\right] + j\frac{4}{9\pi}(ka)^3 E_z{}^0\right\}_0 \quad (52)$$

$$M_x = \frac{8}{3}a^3\left\{H_x{}^0 + \frac{(ka)^2}{30}\left[13H_x{}^0 - \frac{3}{k^2}\frac{\partial^2 H_x{}^0}{\partial z^2} - \frac{2j}{\omega\mu_0}\frac{\partial E_z{}^0}{\partial y}\right] - j\frac{8}{9\pi}(ka)^3 H_x{}^0\right\}_0$$

$$M_y = \frac{8}{3}a^3\left\{H_y{}^0 + \frac{(ka)^2}{30}\left[13H_y{}^0 - \frac{3}{k^2}\frac{\partial^2 H_y{}^0}{\partial z^2} + \frac{2j}{\omega\mu_0}\frac{\partial E_z{}^0}{\partial x}\right] - j\frac{8}{9\pi}(ka)^3 H_y{}^0\right\}_0. \quad (53)$$

The bracketed terms are evaluated at the center of the aperture.

The scattering coefficient t of an aperture is usually defined as the ratio between the energy incident on the aperture and the energy transmitted through the aperture. We obtain thus

$$t = \tfrac{1}{2}\tau \quad (54)$$

where τ is the scattering coefficient of a disk given by (48).

Appendix

Evaluation of Integrals

In the foregoing discussion we need to evaluate some integrals of the form

$$G(x, y) = \int_{\mathrm{Disk}} \frac{f(x', y')}{\pi^2(a^2 - x'^2 - y'^2)^{1/2}((x - x')^2 + (y - y')^2)^{1/2}} dS' \quad (55)$$

where $f(x', y')$ is a polynomial in x' and y'. Bouwkamp [50] has given a solution for this kind of integral in the form:

$$I(n, m, \mu; \rho, \varphi) = \int_0^a \int_0^{2\pi} \frac{P_{2n}^{2m}((a^2 - \rho'^2)^{1/2})((x - x')^2 + (y - y')^2)^{\mu/2}\cos(2m\varphi')}{(a^2 - \rho'^2)^{1/2}((x - x')^2 + (y - y')^2)^{1/2}}\rho' d\rho' d\varphi'$$
$$= \sum_v A_v(n, m, \mu) P_{2n}^{2m}((a^2 - \rho^2)^{1/2})\cos(2m\varphi) \quad (56)$$

TABLE I

$$G(x, y) = \int_0^a \int_0^{2\pi} \frac{f(x', y')\rho' d\rho' d\varphi'}{\pi^2(a^2 - \rho'^2)^{1/2}[(x - x')^2 + (y - y')^2]^{1/2}}$$

$f(x, y)$	$G(x, y)$
1	1
x	$\frac{1}{2} x$
x^2	$\frac{1}{16}[4a^2 + 5x^2 - y^2]$
xy	$\frac{3}{8} xy$
x^3	$\frac{1}{32}[6xa^2 + 7x^3 - 3xy^2]$
x^2y	$\frac{1}{32}[2ya^2 - y^3 + 9x^2y]$
x^4	$\frac{1}{1024}[144a^4 + 144x^2a^2 - 48y^2a^2 + 169x^4 - 102x^2y^2 + 9y^4]$
x^3y	$\frac{1}{1024}[96xya^2 + 210x^3y - 60xy^3]$
x^2y^2	$\frac{1}{1024}[48a^4 + 16x^2a^2 + 16y^2a^2 - 17x^4 + 246x^2y^2 - 17y^4]$

where

$\rho'^2 = x'^2 + y'^2 \leq a^2$

$\rho^2 = x^2 + y^2 \leq a^2$

$0 \leq \varphi \leq 2\pi$

P_{2n}^{2m} = associated Legendre functions

n, m π and μ = integers subject to the condition $\mu \geq 0$, $0 \leq m \leq n$.

The coefficients A_v are:

$$A_v(n, m, \mu) = \frac{\pi(-1)^{n+v}(2v + \frac{1}{2})\Gamma(\mu + 1)\Gamma\left(\frac{\mu + 1}{2}\right)\Gamma\left(\frac{\mu + 1}{2}\right)\Gamma(n + m + \frac{1}{2})\Gamma(v - m + \frac{1}{2})}{\Gamma(n - m + 1)\Gamma(v + m + 1)\Gamma(\frac{1}{2}\mu + n - v + \frac{1}{2})\Gamma(\frac{1}{2}\mu + n - v + 1)\Gamma(\frac{1}{2}\mu - n + v + 1)\Gamma(\frac{1}{2}\mu + n + v + 3/2)} \quad (57)$$

where $\Gamma(n+1) = n! =$ Gamma function

$$\Gamma(\tfrac{1}{2}) = (\pi)^{1/2}.$$

By choosing the integers n, m, and v appropriately and differentiating with respect to x and y the integral (55) can be found for any arbitrary polynomial $f(x, y)$. The calculations involved are, however, quite tedious.

Table I gives $G(x, y)$ for polynomials up to the fourth power in x and y.

Table II lists $\nabla^2 G(x, y)$.

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TABLE II

$f(x, y)$	$\nabla^2 G(x, y)$
1	0
x	0
x^2	$\frac{1}{2}$
xy	0
x^3	$\frac{9}{8} x$
x^2y	$\frac{3}{8} y$
x^4	$\frac{3}{32}[2a^2 + 19x^2 - y^2]$
x^3y	$\frac{225}{256} xy$
x^2y^2	$\frac{1}{32}[2a^2 + 9x^2 + 9y^2]$

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